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Boundary Value Problems for Nonlinear Second-Order Vector Differential Equations

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1. INTRODUCTION

A number of papers have been devoted to the study of different types of boundary value problems for a vector second-order differential equation

$$x'' = f(t, x, x'), \tag{1.1}$$

where $f : [a, b] \times R^n \times R^n \to R^n$ and $x' = dx/dt$. For boundary conditions of the form

$$x(a) = x_a\,, \qquad x(b) = x_b\,, \tag{1.2}$$

with $x_a, x_b \in R^n$ (*Picard* BVP), references can be found in Hartman's book [1]; see also recent papers by K.A. Heimes [2], and J.H. George and W.G. Sutton [3]. For periodic boundary conditions

$$x(a) = x(b), \qquad x'(a) = x'(b) \tag{1.3}$$

(*Poincaré* BVP) references can be found in the papers of H.W. Knobloch [4] and K. Schmitt [5].

The purpose of this work is to show how the author's coincidence degree [6], an extension of Leray–Schauder theory, can be used to give a simple and unified proof of somewhat more general versions of the results of P. Hartman [1, 7], H. W. Knobloch [4], and K. Schmitt [5]. Applications to special vector second-order differential equations are given and a detailed study of the scalar case will be published elsewhere.

Let us also note that a recent paper of A. Lasota and J. A. Yorke [10] uses Leray–Schauder theory to generalize an existence theorem of P. Hartman for the Picard BVP distinct of the one extended here.

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Moreover, since the time where this paper was submitted for publication, a number of related results have appeared in the literature. First, Corollary 6.2 of the present paper, in the periodic case, has been proved independantly by Bebernes and Schmitt [11, 12], together with other results, via one theorem of Krasnosel'skii [13]. A direct proof of this corollary, by an argument similar to the one used here, can also be found in the joint book [14] of Rouche and the author (Chapter XI, Section 5). To end with the periodic case, let us also note that Bebernes [15] has recently shown how to use Leray–Schauder continuation theorem to prove results in the line of this paper and Knobloch's one quoted above.

Concerning other boundary value problems for (1.1), the author [16] has used coincidence degree theory to prove that the conclusions of the Theorem 6.1 of this paper still hold for the boundary conditions

$$x(a) = x_a , \qquad x'(b) = 0, \tag{1.4}$$

and

$$x'(a) = x'(b) = 0, \tag{1.5}$$

with the same assumptions than in the periodic case for (1.5) and with (6.1) replaced by

$$V(x_a) < 0$$

for (1.4). Moreover, the same technique is applied in [16] to mixed boundary conditions of the form

$$Ax(a) - A'x'(a) = Bx(b) + B'x(b) = 0, \tag{1.6}$$

where A, A', B, B' are diagonal $(n \times n)$-matrices with non-negative elements such that $A + A'$ and $B + B'$ are positive definite. It is proved that Corollary 6.2 of this paper still holds for the boundary conditions (1.6), which implies, as a special case, some results of Corduneanu [17].

2. A Basic Existence Theorem for Nonlinear Operator Equations

X and Z being locally convex separated topological vector spaces, let $L : \operatorname{Dom} L \subset X \to Z$ be a (not necessarily continuous) Fredholm mapping of index zero and $N : \operatorname{Dom} N \subset X \to Z$ a (not necessarily linear) continuous mapping and let us choose an orientation upon $\operatorname{Ker} L$ and Coker L. Then, if $\Omega \subset X$ is a finitely bounded open set with closure $\operatorname{cl} \Omega$ and boundary $\operatorname{bdry} \Omega$ such that:

(i) $\operatorname{cl} \Omega \subset \operatorname{Dom} N$ and $\operatorname{cl} \Omega \cap \operatorname{Dom} L \neq \varnothing$;

(ii) either N is completely continuous on $\operatorname{cl} \Omega$ and L has a continuous right inverse on $\operatorname{Im} L$ or $N(\operatorname{cl} \Omega)$ is bounded and L has a compact right inverse on $\operatorname{Im} L$;

(iii) $0 \notin (L - N)(\operatorname{bdry} \Omega \cap \operatorname{Dom} L)$;

it is possible to define for the couple (L, N) an integer, the $\mathscr{L}^+$-coincidence degree in Ω $d[(L, N), \Omega]$ such that, if $X = Z$ and $L = I$, the identity,

$$d[(L, N), \Omega] = d_{LS}[I - N, \Omega, 0]$$

(the Leray–Schauder degree at zero of $I - N$ in Ω) [6]. This coicidence degree conserves most of the basic properties of Leray–Schauder degree and furnishes a natural proof of the following existence theorem (see [6]):

PROPOSITION 2.1. *If assumptions* (i) *and* (ii) *above are satisfied and, y being a fixed element of* $\operatorname{Im} L$, *if the following conditions are verified*:

(a) $Lx \neq \lambda Nx + y$ *for every* $x \in \operatorname{bdry} \Omega \cap \operatorname{Dom} L$ *and every* $\lambda \in]0,1[$;

(b) $L^{-1}y \in \Omega$ *or* $QN(c + Ky) \neq 0$ *for every* $c \in \operatorname{bdry}(-Ky + \Omega) \cap \operatorname{Ker} L$ *depending on whether* $\operatorname{Ker} L = \{0\}$ *or* $\operatorname{Ker} L \neq \{0\}$, *with Q any continuous projector on Z such that* $Z = \operatorname{Im} Q \oplus \operatorname{Im} L$ *and K a right inverse of L mapping* $\operatorname{Im} L$ *into a topological complement of* $\operatorname{Ker} L$ *in* X;

(c) *if* $\operatorname{Ker} L \neq \{0\}$, *the Brouwer degree*

$$d_B[n_0, (-Ky + \Omega) \cap \operatorname{Ker} L, 0]$$

is not zero, where $n_0 : (-Ky + \operatorname{cl} \Omega) \cap \operatorname{Ker} L \to \operatorname{Ker} L$ *is the mapping defined by*

$$n_0(c) = -JQN(c + Ky)$$

with $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *any isomorphism, then, for every* $\lambda \in [0, 1]$, *the equation*

$$Lx = \lambda Nx + y \tag{2.1}$$

has at least one solution in $\operatorname{cl} \Omega$ *and hence the same is true for the equation*

$$Lx = Nx + y. \tag{2.2}$$

Note that this result is an extension to the case where $\operatorname{Ker} L \neq \{0\}$ of a well-known existence theorem for the case where L^{-1} exists.

3. Basic Assumptions and Reduction to Equation (2.2)

Let us consider the second-order vector differential equation

$$x'' = f(t, x, x'), \tag{3.1}$$

where $f : [a, b] \times R^n \times R^n \to R^n$ is continuous and $a < b$. If x_a, x_b are fixed elements of R^n, a solution of the *Picard boundary value problem* (PdBVP, for short) for equation (3.1) will be a mapping $x : [a, b] \to R^n$ of class C^2 which satisfies (3.1) and the boundary conditions

$$x(a) = x_a, \qquad x(b) = x_b. \tag{3.2}$$

A solution of the Poincaré boundary value problem or periodic boundary value problem (PBVP, for short) for equation (3.1) will be a mapping x: $[a, b] \to R^n$ of class C^2 which satisfies (3.1) and the boundary conditions

$$x(a) = x(b), \qquad x'(a) = x'(b). \tag{3.3}$$

Now, $|\cdot|$ being the Euclidean norm in R^n, let $B = C^1([a, b], R^n)$ be the Banach space of mappings $x : [a, b] \to R^n$ of class C^1 with the norm

$$\| x \| = \max\{ \sup_{t\in[a,b]} | x(t)|, \sup_{t\in[a,b]} | x'(t)|\}$$

and let $\tilde{B}$ be the subspace of B defined by

$$\tilde{B} = \{x \in B : x(a) = x(b), \quad x'(a) = x'(b)\}$$

with the induced norm. It is very easy to verify the following

Proposition 3.1. *The* PdBVP *for equation* (3.1) *can be written in the form* (2.2) *if we take*

$$X = \operatorname{Dom} N = B,\ Z = B \times R^n \times R^n,\ \operatorname{Dom} L = \{x \in B : x \text{ is of class } C^2\},$$
$$L : x \mapsto [x'', x(a), x(b)], \quad N : x \mapsto [f(\cdot, x(\cdot), x'(\cdot)), 0, 0],\ y = (0, x_a, x_b).$$

Moreover, $\operatorname{Ker} L = \{0\}$, $\operatorname{Im} L = Z$, L^{-1} *is compact and* N *is continuous and takes bounded sets into bounded sets.*

Proposition 3.2. *The* PBVP *for equation* (3.1) *can be written in the form* (2.2) *if we take*

$$X = \operatorname{Dom} N = \tilde{B},\ Z = B,\ \operatorname{Dom} L = \{x \in \tilde{B} : x \textit{ is of class } C^2\},$$
$$L : x \mapsto x'',\ N : x \mapsto f(\cdot, x(\cdot), x'(\cdot)),\ y = 0.$$

Moreover,

$$\operatorname{Ker} L = \{x \in \tilde{B} : x \text{ is a constant mapping}\}$$
$$= \left\{x \in \tilde{B} : x(t) = (b-a)^{-1} \int_a^b x(s)\, ds,\ \forall t \in [a, b]\right\},$$
$$\operatorname{Im} L = \left\{x \in B : \int_a^b x(s)\, ds = 0\right\}$$

and hence a projector Q in B such that $B = \operatorname{Im} Q \oplus \operatorname{Im} L$ is given by

$$Q: x \mapsto (b-a)^{-1} \int_a^b x(s)\, ds,$$

which yields $\operatorname{Im} Q = \operatorname{Ker} L$. Lastly, L has a compact right inverse and N is continuous and takes bounded sets into bounded sets.

Note that the compactness conclusions in the two propositions above are obtained as usual by an easy application of Arzela–Ascoli theorem.

4. The Concept of Bounding Function Relative to Equation (3.1)

Let us introduce the following

Definition 4.1. A *bounding function relative to equation* (3.1) is a function $V : R^n \to R$ of class C^2 which satisfies the following conditions:

(a) the set

$$\Phi = \{x \in R^n : V(x) < 0\}$$

is bounded;

(b) if $u : R^n \to R^n$ and $W : R^n \to \mathscr{L}(R^n, R^n)$ denote, respectively, the gradient vector and the Hessian matrix functions of V, then $\langle\, ,\rangle$ being the scalar product in R^n,

$$\langle W(x)y, y\rangle + \lambda\langle u(x), f(t, x, y)\rangle > 0 \tag{4.1}$$

for every $(t, x, y, \lambda) \in [a, b] \times R^n \times R^n \times]0, 1[$ such that

$$V(x) = 0 \quad \text{and} \quad \langle u(x), y\rangle = 0.$$

The name "bounding function" follows from the fact that the existence of such a function implies the existence of an "*a priori bound*" for solutions of the PdBVP and the PBVP for every equation of the family

$$x'' = \lambda f(t, x, x') \tag{4.2}$$

with $\lambda \in]0, 1[$ as follows from

PROPOSITION 4.1. *Let us suppose that there exists a bounding function relative to equation* (3.1). *Then, for every* $\lambda \in]0, 1[$, *each possible solution* x *of the* PBVP *for* (4.2) *is such that*

$$V[x(t)] < 0 \quad \forall\, t \in [a, b], \tag{4.3}$$

or

$$V[x(\tau)] > 0 \text{ for some } \tau \in [a, b], \tag{4.4}$$

and, if $x_a, x_b \in R^n$ *satisfy the relations*

$$V(x_a) < 0, \qquad V(x_b) < 0, \tag{4.5}$$

the same is true for the corresponding PdBVP.

Proof. Let $\lambda \in]0, 1[$ and, x being a possible solution of the PdBVP or PBVP for equation (4.2), let us define the function $v : [a, b] \to R$ by

$$v(t) = V[x(t)], \qquad t \in [a, b].$$

Then v is of class C^2 in $[a, b]$ and

$$v'(t) = \langle u[x(t)], \quad x'(t)\rangle \tag{4.6}$$

$$v''(t) = \langle W[x(t)]\, x'(t), \quad x'(t)\rangle + \lambda\langle u[x(t)], \quad f[t, x(t), x'(t)]\rangle. \tag{4.7}$$

If $v(t)$ has no negative absolute maximum in $[a, b]$, there will exist a $\tau \in [a, b]$ such that $v(\tau) > 0$ and the Lemma is proved. If $v(t)$ has a negative absolute maximum which is attained at $s \in]a, b[$, then

$$v(s) \leqslant 0, \qquad v'(s) = 0, \qquad v''(s) \leqslant 0,$$

and it follows immediately from (4.6), (4.7), and the definition of a bounding function that $v(s) < 0$, and hence

$$V[x(t)] = v(t) \leqslant v(s) < 0\ \forall\, t \in [a, b].$$

If the negative absolute maximum of v is attained at a or b, then it follows immediately from (4.5) that $V[x(t)] < 0\ \forall\, t \in [a, b]$ in the PdBVP case. For the PBVP, we then have necessarily

$$v(t) \leqslant v(a) = v(b) \leqslant 0\ \forall\, t \in [a, b],$$

which immediately implies that

$$0 \leqslant \langle u[x(b)], x'(b)\rangle = \langle u[x(a)], x'(a)\rangle \leqslant 0. \tag{4.8}$$

But, then, $v(a) = v(b) < 0$ because, if $v(a) = v(b) = 0$, it follows from (4.6), (4.7), (4.8), and the fact that V is a bounding function that $v(t) > v(a)$ in some neighborhood of a and $v(t) > v(b)$ in some neighborhood of b, a contradiction. Thus Proposition 4.1 is completely proved.

A particular bounding function has been used by P. Hartman [7] in his existence theorem for the PdBVP relative to (3.1) in which he introduces the assumption:

There exists $R > 0$ such that

$$|y|^2 + \langle x, f(t, x, y)\rangle > 0, \tag{4.9}$$

for every $(t, x, y) \in [a, b] \times R^n \times R^n$ such that $|x| = R$ and $\langle x, y\rangle = 0$. In this case, the function $V: x \mapsto |x|^2 - R^2$ is a bounding function relative to (3.1) because (4.9) clearly implies that

$$|y|^2 + \lambda\langle x, f(t, x, y)\rangle > 0,$$

for every $(t, x, y, \lambda) \in]a, b[\times R^n \times R^n \times]0, 1[$ such that $|x| = R$ and $\langle x, y\rangle = 0$.

Let us also note that, in their existence theorems, H.W. Knobloch [4] and J.H. George and W.G. Sutton [3] have used auxiliary functions allowed to depend also on t or on t and x' but, on the other side, submitted to more restrictive supplementary conditions than our bounding functions.

5. The Concept of Nagumo Equation with Respect to a Bounding Function and a Boundary Value Problem

Let us introduce the following

Definition 5.1. If V is a bounding function relative to (3.1), this equation will be called a *Nagumo equation with respect to V and a given boundary value problem* if, for each $\lambda \in]0, 1[$, every possible solution x of this boundary value problem for equation (4.2) which satisfies the relation

$$V[x(t)] \leqslant 0, \qquad \forall\, t \in [a, b],$$

is such that

$$|x'(t)| < S, \qquad \forall\, t \in [a, b]$$

with $S > 0$ independant of λ.

The name Nagumo equation follows from the fact that in the scalar case and for $V(x) = |x|^2 - R^2$, a sufficient condition for (3.1) to be a Nagumo equation was first introduced by M. Nagumo [8]. It has been extended to the vector case by P. Hartman [7] in the following way

Nagumo–Hartman condition. If the following conditions hold:

(i) *there exists a (Nagumo) function* $\varphi : R_+ \to R_+$ *such that*

$$\int^{\infty} \frac{s\,ds}{\varphi(s)} = \infty$$

and

$$|f(t, x, y)| \leqslant \varphi(|y|)$$

in

$$E(R) = \{(t, x, y) \colon t \in [a, b], \qquad |x| \leqslant R, \qquad y \in R^n\}$$

with $R > 0$;

(ii) *when* $n > 1$,

$$|f(t, x, y)| \leqslant 2\alpha[\langle x, f(t, x, y)\rangle + |y|^2] + \kappa$$

in $E(R)$, *where* $\alpha \geqslant 0$ *and* $\kappa \geqslant 0$ *are constants, then every solution* x *of equation* (3.1) *which satisfies*

$$|x(t)| \leqslant R, \qquad \forall\, t \in [a, b],$$

is such that

$$|x'(t)| < S, \qquad \forall\, t \in [a, b],$$

where S *depends only on* $\varphi(s)$, α, R, κ *and* $b - a$.

For a proof, see for example [1], pp. 428–431.

An easy consequence is the following

PROPOSITION 5.1. *If equation* (3.1) *satisfies the Nagumo-Hartman condition, then, for every bounding function* V *relative to* (3.1) *such that*

$$\Phi \subset B(R),$$

the closed ball of center zero and radius R, (3.1) *is a Nagumo equation with respect to* V *and the* PdBVP *or the* PBVP.

Proof. The proof follows at once from the fact that if the right-hand side of (3.1) verifies a Nagumo–Hartman condition, the same is true, with the same functions and constants, for the right-hand side of equation (4.2) for every $\lambda \in]0, 1[$.

But (3.1) can be a Nagumo equation without verifying the Nagumo–Hartman condition as follows from

PROPOSITION 5.2. *If* $g : R^n \to R$ *is class* C^1 *and* $h : [a, b] \times R^n \to R^n$ *is continuous, then the equation*

$$x'' = \operatorname{grad} g(x') + h(t, x) \tag{5.1}$$

is a Nagumo equation with respect to any bounding function V *and the* PBVP.

Proof. If $\lambda \in]0, 1[$ and if x is any possible solution of the PBVP for equation

$$x'' = \lambda[\operatorname{grad} g(x') + h(t, x)],$$

then we have

$$(b-a)^{-1} \int_a^b \langle x''(t), x''(t)\rangle \, dt = (b-a)^{-1} \int_a^b \langle \operatorname{grad} g[x'(t)], x''(t)\rangle \, dt$$
$$+ (b-a)^{-1} \int_a^b \langle h[t, x(t)], x''(t)\rangle \, dt,$$

and hence, using the periodicity of x' and Schwarz inequality,

$$\left[(b-a)^{-1} \int_a^b |x''(t)|^2 \, dt\right]^{1/2} < H \tag{5.2}$$

with

$$H > \sup_{\substack{|x| \leqslant R \\ t \in [a,b]}} |h(t, x)|,$$

and $R > 0$ such that the set Φ corresponding to V is contained in the closed ball of center zero and radius R. From (5.2) and a well known inequality we obtain then

$$|x'(t)| < (b-a)H, \qquad \forall\, t \in [a, b]$$

and Proposition 5.2 is proved.

6. The Basic Existence Theorem for PdBVP and PBVP and Applications

We can now use the results and concepts of the preceding sections to prove the following basic

THEOREM 6.1. *If there exists a bounding function V relative to* (3.1) *for which* (3.1) *is a Nagumo equation for the* PdBVP (*resp.* PBVP) *and if*

$$V\{(b-a)^{-1}[(t-a)\, x_b + (b-t)\, x_a]\} < 0 \;\forall\, t \in [a, b] \tag{6.1}$$

(*resp. if*

$$d_B[\operatorname{grad} V, \Phi, 0] \neq 0) \tag{6.2}$$

then, for every $\lambda \in [0, 1]$, the PdBVP (*resp.* PBVP) *for equation* (4.2) *has at least one solution x such that*

$$V[x(t)] \leqslant 0 \;\forall\, t \in [a, b],$$

and the same is true for equation (3.1).

Proof. Let us introduce the open bounded set

$$\Omega = \{x \in X : V[x(t)] < 0, \forall t \in [a, b], \sup_{t \in [a,b]} | x'(t)| < S\}$$

where $X = B$ or $\tilde{B}$ depending on whether we consider the PdBVP or the PBVP, and S is given by definition 5.1. Using Propositions 3.1, 3.2, 4.1, and Definition 5.1, we see that assumption (a) of Proposition 2.1 is satisfied both for the PdBVP and the PBVP for the set Ω defined above. Now, (6.1) implies that condition (b) of Proposition 2.1 is satisfied (we are in the case where $\operatorname{Ker} L = \{0\}$) and hence Theorem 6.1 is proved for the PdBVP. In the periodic case, we have

$$n_0(c) \equiv QN(c) = (b - a)^{-1} \int_a^b f(t, c, 0)\, dt, \qquad c \in R^n$$

(we have used the natural isomorphism between $\operatorname{Ker} L$ and R^n) and it follows easily from (4.1) with $y = 0$ that

$$\left\langle \operatorname{grad} V(c), (b - a)^{-1} \int_a^b f(t, c, 0,)\, dt \right\rangle > 0, \tag{6.3}$$

for every $c \in R^n$ such that $V(c) = 0$ and hence for every

$$c \in \operatorname{bdry} \Phi = \operatorname{Ker} L \cap \operatorname{bdry} \Omega.$$

Thus, condition (b) of Proposition 2.1 is satisfied in the PBVP case and moreover, using (6.3), the Poincaré–Bohl theorem of Brouwer degree theory and condition (6.2) we find

$$d_B[n_0, \Omega \cap \operatorname{Ker} L, 0] = d_B[\operatorname{grad} V, \Phi, 0] \neq 0.$$

Thus, condition (c) of Proposition 2.1 is verified and Theorem 6.1 is proved for the PBVP.

COROLLARY 6.1. *Theorem* 6.1 *still holds if the strict inequality in conditions* (4.1) *and* (6.1) *is replaced by a nonstrict one.*

Proof. The proof follows by exactly the same approximation and compactness argument as used in [1], p. 433, last paragraph.

COROLLARY 6.2. *If there exists* $R > 0$ *such that*

$$| y |^2 + \langle x, f(t, x, y)\rangle \geqslant 0 \tag{6.4}$$

for every $(t, x, y) \in [a, b] \times R^n \times R^n$ *verifying* $| x | = R$ *and* $\langle x, y\rangle = 0$, *and if* (3.1) *is a Nagumo equation with respect to* $V(x) = | x |^2 - R^2$ *and the*

PdBVP *with* $|x_a|$, $|x_b| \leqslant R$ (*resp. the* PBVP), *then the* PdBVP (*resp.* PBVP) *for* (3.1) *has at least one solution* x *such that*

$$|x(t)| \leqslant R, \qquad \forall\, t \in [a, b].$$

Proof. It suffices to take the bounding function $V(x) = |x|^2 - R^2$ and to use Corollary 6.1.

When the condition that (3.1) to be a Nagumo equation is replaced by the Nagumo–Hartman condition, Corollary 4.2 was first proved in the PdBVP case, in a different way, by P. Hartman [7] and, in the PBVP case, under different types of more restrictive conditions, by H. W. Knobloch [4] and K. Schmitt [5].

COROLLARY 6.3. *Let* g *and* h *be as in Proposition* 5.2 *and let* $k : R^n \to R^n$ *be a globally Lipschitzian mapping. If* grad $g(y)$ (*resp.* $k(y)$) *has either the same or the opposite direction of* y *and if there exists* $R > 0$ *such that*

$$\langle x, h(t, x)\rangle \geqslant 0 \tag{6.5}$$

for every $(t, x) \in [a, b] \times R^n$ *such that* $|x| = R$, *then the* PBVP *for equation*

$$x'' = \operatorname{grad} g(x') + h(t, x) \tag{6.6}$$

$$(\text{resp. } x'' = k(x') + h(t, x)) \tag{6.7}$$

has at least one solution x *such that* $|x(t)| \leqslant R$ *for every* $t \in [a, b]$.

Proof. For equations (6.6) and (6.7), condition (6.4) of Corollary 6.2 follows from (6.5) because

$$\langle \operatorname{grad} g(y), x\rangle = \langle k(y), x\rangle = 0$$

if $\langle x, y\rangle = 0$. The global Lipschitz condition imposed on k implies that (6.7) satisfies a Nagumo–Hartman condition (see [1], p. 433); on the other hand, it follows from Proposition 5.2 that (6.6) is a Nagumo equation with respect to the PBVP and the bounding function $V(x) = |x|^2 - R^2$. The result then follows from Corollary 6.2.

In the case of equation (6.7) with $h(t, x) = Ax$, A a positive definite matrix, Corollary 6.3 has been proved in a different way by K. Schmitt [5] under the supplementary condition

$$q \leqslant 2\mu^{1/2},$$

where q is the Lipschitz constant of the mapping k, and μ the least eigenvalue of matrix A.

COROLLARY 6.4. *For each* $i = 1,\ldots, n$, *let* $\alpha_i(t, x)$ *and* $\beta_i(t, x)$ *be continuous functions on* $[a, b] \times R^n$ *into* R *having the properties that there exist constants* $\delta_i > 0$ *and* $\Delta_i > 0$ *such that* $\alpha_i(t, x) \geqslant \delta_i$ *and* $|\beta_i(t, x)| \leqslant \Delta_i$ $(i = 1,\ldots, n)$ *for every* $(t, x) \in [a, b] \times R^n$. *Then the* PBVP *for the system*

$$x_i'' = x_i \alpha_i(t, x) + \beta(t, x) \qquad (i = 1,\ldots, n) \tag{6.8}$$

has at least one solution.

Proof. The system (6.8) clearly satisfies a Nagumo–Hartman condition. On the other hand, if

$$\delta = \min(\delta_1, \ldots, \delta_n), \qquad \Delta = \left(\sum_{i=1}^{n} \Delta_i^2\right)^{1/2},$$

then for every $R \geqslant \Delta/\delta$, the function

$$V(x) = x^2 - R^2$$

is a bounding function for (6.8) because

$$\sum_{i=1}^{n} [x_i^2 \alpha_i(t, x) + x_i \beta_i(t, x)] \geqslant \delta \mid x \mid^2 - \Delta \mid x \mid = \delta \mid x \mid (\mid x \mid - \Delta/\delta).$$

The result then follows from Corollary 6.2.

This corollary has been proved in a different way by K. Schmitt [5]. For a more general existence theorem concerning the PBVP for a more general system of type (6.8), proved directly from the argument used in Proposition 2.1, see [9].

REFERENCES

1. P. HARTMAN, "Ordinary Differential Equations," Wiley, New York, 1964.
2. K. A. HEIMES, Boundary value problems for ordinary nonlinear second-order systems, *J. Diff. Eq.* **2** (1966), 449–463.
3. J. H. GEORGE AND W. G. SUTTON, Application of Liapunov theory to boundary value problems, *Proc. Amer. Math. Soc.* **25** (1970), 666–671.
4. H. W. KNOBLOCH, On the existence of periodic solutions for second order vector differential equations, *J. Diff. Eq.* **9** (1971), 67–85.
5. K. SCHMITT, Periodic solutions of systems of second-order differential equations, *J. Diff. Eq.* **11** (1972), 180–192.
6. J. MAWHIN, Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces, *J. Diff. Eq.* **12** (1972), 610–636.
7. P. HARTMAN, On boundary value problems for systems of ordinary, nonlinear, second-order differential equations, *Trans. Amer. Math. Soc.* **96** (1960), 493–509.

8. M. NAGUMO, Ueber die Differentialgleichung $y'' = f(x, y, y')$, *Proc. Phys.-Math. Soc. Japan*, Ser. 3 **19** (1937), 861–866.
9. J. MAWHIN, Existence of periodic solutions for higher-order differential systems that are not of class D, *J. Diff. Eq.* **8** (1970), 523–530.
10. A. LASOTA AND J. A. YORKE, Existence of solutions of two-point boundary value problems for nonlinear systems, *J. Diff. Eq.* **11** (1972), 509–518.
11. J. W. BEBERNES AND K. SCHMITT, Periodic boundary value problems for systems of second order differential equations, *J. Diff. Eq.* **13** (1973), 32–47.
12. J. W. BEBERNES AND K. SCHMITT, An existence theorem for periodic boundary value problems for systems of second-order differential equations, *Arch. Math. (Brno)*, to appear.
13. M. A. KRASNOSEL'SKII, "Translation along Trajectories of Differential Equations," *Transl. of Russian Monographs Ser.*, Vol. **19**, Amer. Math. Soc., Providence, Rhode Island, 1968.
14. N. ROUCHE AND J. MAWHIN, "Equations différentielles ordinaires. Tome 2: Stabilité et Solutions Périodiques," Masson, Paris, 1973.
15. J. W. BEBERNES, A simple alternative problem for finding periodic solutions of second-order ordinary differential systems, *Proc. Amer. Math. Soc.*, to appear.
16. J. MAWHIN, Degré de coïncidence et problèmes aux limites pour des équations différentielles ordinaires et fonctionnelles, *Sémin. Math. Appl. Méc. Univ. Louvain, Rapport* n° 64, Vander, Louvain, 1973.
17. C. CORDUNEANU, Sopra i problemi ai limiti per alcuni sistemi di equazioni differenziali non lineari, *Rend. Acc. Sci. Fiz. Mat. Napoli* (4), **25** (1958), 98–106.